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SOLVABILITY OF A NORMAL SUBGROUP IN RELATION TO ITS CHARACTER DEGREES

***M. A GANIYU[1], F. M JIMOH[1], A. D. AKWU[2]**

[*1]Department of Physical Science , Al-hikmah University, Ilorin, Kwara State, Nigeria.
Department of Mathematics, Federal University of Agriculture, Makurdi, Benue State.

***Corresponding author**: bidex1425@yahoo.com **Tel:** +2348051711777

ABSTRACT

In this work, how the structure of a normal subgroup of a group G is influenced by the degrees of an appropriate subset of irreducible character of a group G was verified. The characters that were used in controlling the structure of N Δ G are exactly those whose kernels do not contain N.
Given that N Δ G,

Irr (G/N) = { χ Є Irr (G)/N $\not\subseteq$ ker (χ) }
and

cd ($^G/_N$) = { χ (1) / χ $\in$ Irr ($^G/_N$)}

Keywords: Normal Subgroup, Character degrees, Solvable groups, Derived length and irreducible Character.

INTRODUCTION

In group theory, the character of a group representation is a function on the group which associates to each group element, the trace of the corresponding matrix. The character carries the essential information about the representation in a more condensed form.

Let V be a finite dimensional vector space over a field F and let ϱ : G ® GL (V) be a representation of a group G on V. The character of ϱ is the function.

χ : G ® F given by

χ (g) = Tr ((g)). Where Tr is the trace.

A character χ is called irreducible if ϱ is an irreducible representation. A character χ is linear if the dimension of ϱ is 1. If χ is a character of G. then the kernel of χ is given by:

Ker χ = (g $\in$ G: χ (g) = χ (1)

Characters are class functions i.e. they take a constant value on a given conjugacy class. Isomorphic representations have the same characters and if a representation is the direct sum of subrepresentations, then the corresponding character is the sum of the char-

acters of those subrepresentations.
Let P and s be representations of G, then the following identities hold:

χ_{ϱ} Ås = χ_{ϱ} + χ s

χ_{ϱ} Äs = χ_{ϱ} . χ s

$\chi_{\varrho} = \bar{\chi}_{\rho}$

χ_{Alt^2} ρ (g) = ½ [(χ_ρ (g))^2 - χ_ρ (g^2)]

χ_{sym^2} ρ (g) = ½ χ_ρ (g))^2 + χ_ρ (g^2)]

Where ρ Äs is the direct sum, ρ Äs is the tensor product, ρ * denotes the conjugate transpose of ρ, Alt^2 is the alternating product and sym^2 is the symmetric square.

Garrison (1973) wrote on 'on groups with a small number of character degrees' where he stated that if |cd(G)|= 4, then dl (G) ≤ |cd (G)| for all solvable groups. Isaacs (1975) also stated that if |cd (G)| ≤ 3, then G is necessarily solvable and dl (G) ≤ |cd (G)| in his work character degrees and derived length of a solvable group.

Berger (1976) 'characters and derived length in groups of odd order' wrote that if |G| is odd, then dl (G) ≤ |cd (G)|. Also, Gluck (1985) wrote on Bounding the number of character degrees of a solvable groups where he stated that dl (G) ≤ 2/cd(G)/ holds for all solvable group.

Mark (1998) wrote on derived lengths and character degrees. Gustavo and Alexander (2001) treated groups with two extreme character degrees and their normal subgroups. Isaacs and Moreto (2001) established a linkage between the character degrees and Nilpotency class of a P-group.

Alexander and Sanius (2005) wrote on character degrees, blocks and normal subgroup. Chen et al (2006) worked on groups with character degrees of two distinct primes. Cossey (2006) showed the bounds on the number of lifts of a Brauer Character in a P-solvable group.

The goal of this paper is to verify how the structure of a normal subgroup of G is influenced by the degrees of an appropriate subset of Irr (G).

RESULTS AND DISCUSSION

Character table

The irreducible complex characters of a finite group form a character table which encodes much useful information about the group G in a compact form. Each row is labelled by an irreducible character and the entries in the row are the values of that character on the representatives of the respective conjugacy class of G. The columns are labelled by (representatives of) the conjugacy classes of G. It is customary to label the first row by the trivial character and the first column by (the conjugacy class of) the identity. The entries of the 1st column are the values of the irreducible characters at the identity, the degrees of the irreducible characters. Characters of degree are known as Linear Character.

The character table is always square because the number of irreducible representations is equal to the number of conjugacy classes. The first row of the character table always consist of 1's and that corresponds to the trivial representation. The order of G is given by the sum of the squares of the entries of the 1st column (the degrees of the irreducible characters). More generally, the sum of the squares of the absolute values of the entries in any column gives the order of the centralizer of an element of the corresponding conjugacy classes.

All normal subgroups of G (and whether or not G is simple) can be recognised from its character table. The kernel of a character χ is the set of elements g in G for which χ (g) = χ (1). This is a normal subgroup of G.

Orthogonality relations

The space of complex – valued class functions of a finite group G has a natural inner product.

$$< \square , \beta > = \frac{1}{(G)} \sum_{g \in G} \square (g) \, \beta \overline{(g)}$$

Where $\beta \overline{(g)}$ means the complex conjugate of the value of β on g. With respect to this product, the irreducible characters form an orthonormal basis for the space of class functions, and this yield the orthogonality relation for the rows of the character table.

$$< \chi_i, \chi_j > = \begin{cases} 0 & if \quad i \neq j \\ 1 & if \quad i = j \end{cases}$$

For g, h ∈ G, the orthogonality relation for column is as follows:

$$\sum_{\chi i} \chi_i (g) \overline{\chi i(h)} = (\square|C_G (g).|@ \; 0,)$$ if g, h are conjugate

otherwise

Where the sum is overall of the irreducible characters χ_i of G and the symbol $|C_G(g)|$ denotes the order of the centralizer of g. The orthogonality relations can aid many computations including: decomposing an unknown character as a linear combination of irreducible characters; constructing the complete character table when only some of the irreducible characters are known; finding the orders of the centralizers of representatives of the conjugacy classes of a group G; Finding the order of the group. *Theorem 1: Berkovich's Theorem [3]*

Let N ◅ G and suppose that every member of cd (G/N^1) is divisible by some fixed prime number P. Then N is solvable and has a normal P – complement.

Verification of Berkovich's Theorem

Let S_4 (a symmetic group on 4 objects) be a finite group of order 24. i.e. $|S_4|$ = 24. The elements of S_4 include:

S_4 = { $\begin{pmatrix}1234\\1234\end{pmatrix}$, $\begin{pmatrix}1234\\1243\end{pmatrix}$, $\begin{pmatrix}1234\\1324\end{pmatrix}$, $\begin{pmatrix}1234\\1342\end{pmatrix}$, $\begin{pmatrix}1234\\1424\end{pmatrix}$, $\begin{pmatrix}1234\\1432\end{pmatrix}$, $\begin{pmatrix}1234\\2134\end{pmatrix}$, $\begin{pmatrix}1234\\2143\end{pmatrix}$, $\begin{pmatrix}1234\\2314\end{pmatrix}$, $\begin{pmatrix}1234\\2341\end{pmatrix}$

(1) (34) (23) (234) (243) (24) (12) (12)(34) (123) (1234)

$\begin{pmatrix}1234\\2413\end{pmatrix}$, $\begin{pmatrix}1234\\2431\end{pmatrix}$, $\begin{pmatrix}1234\\3124\end{pmatrix}$, $\begin{pmatrix}1234\\3142\end{pmatrix}$, $\begin{pmatrix}1234\\3214\end{pmatrix}$, $\begin{pmatrix}1234\\3241\end{pmatrix}$, $\begin{pmatrix}1234\\3412\end{pmatrix}$,

$\begin{pmatrix}1234\\3421\end{pmatrix}$, $\begin{pmatrix}1234\\4123\end{pmatrix}$, $\begin{pmatrix}1234\\4132\end{pmatrix}$

(1234) (124) (132) (1342) (13) (134) (13) (24) (1324) (1432) (142)

$\begin{pmatrix}1234\\4213\end{pmatrix}$ $\begin{pmatrix}1234\\4231\end{pmatrix}$ $\begin{pmatrix}1234\\4312\end{pmatrix}$ $\begin{pmatrix}1234\\4321\end{pmatrix}$

(143) (14) (1423) (14)(23)}

The set of all even permutations form a group A_4 of order 12 which include.

A^1_4 = { (1) , (12) (34) , (13)(24) , (14)(23) (123) , (124), (132) , (134), (142) (143) (234) (243)

Which form a normal subgroup of S_4. The commutator subgoup of A^1_4 was obtained by using the formular

$A^1_4 = \{[\chi, y] : \chi^{-1} y^{-1} \chi \; y \in A_4\}$

A^1_4 = {(1), (12) (34), (13 (24), (14) (23)} which is called four group or (klein 4-group).

To get the character table of S_4; we let (1), (12), (12)(34), (1234), (123) be representatives of its conjugacy classes. This implies that S_4 have 5 irreducible characters denoted by χ_1 χ_2 χ_3 χ_4 χ_5

Table 1: Character table of S_4

Representative	(1)	(12)	(12) (34)	(1234)	(123)
Class Size	1	6	3	6	8
|CG(g)|	24	4	8	4	3
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	2	0	2	0	-1
χ_4	3	1	-1	-1	0
χ_5	3	-1	-1	1	0

By definition

Irr (S_4/A^1_4) = { $\chi \in$ Irr (S_4)/$A^1_4 \not\subseteq$ ker (χ) }

Irr (S_4/A^1_4) = { χ_4 , χ_5 }

And cd (S_4/A^1_4) = { χ (1)/ $\chi \in$ Irr (S_4/A^1_4) }

Þ cd (S_4/A^1_4) = { χ_4 (1), χ_5 (1) }
Þ cd (S_4/A^1_4) = { 3, 3, }

Suppose every member of cd (S_4/A^1_4) is divisible by a fixed prime 3; then A_4 is solvable which is true and A_4 has a normal P-Complement which means A_4 has a normal subgroup of index P.
i.e A^1_4 is a normal subgroup of A_4 and [A_4 : A^1_4] = $\frac{|A_4|}{A^1_4}$ = $\frac{12}{4}$ = 3, the fixed prime

Theorem 2: Isaacs and Greg Theorem [10]

Let N Δ G and suppose that |cd (G/N)| ≤ 1, then dl (N) ≤ |cd (G/N) | and in particular, N is abelian.

Verification:

Let $C_4 \times C_2$ be finite group of order 8 where
C_4 = { 1, a, a^2, a^3}, a^4 = 1
and C_2 = {1, b}, b^2 = 1

$C_4 \times C_2$ = { (1, 1), (1, b), (a, 1), (a, b), (a^2, 1), (a^2, b), (a^3, 1), (a^3, b)}

Order of each element in $C_4 \times C_2$ include:

(1, 1)	-	1
(1, b)	-	2
(a, 1)	-	4
(a, b)	-	4
(a^2, 1)	-	2
(a^2, b)	-	2
(a^3, 1)	-	4
(a^3, b)	-	4

Subgroup of $C_4 \times C_2$ of order one
H_1 = {(1,1)}

Subgroup of $C_4 \times C_2$ of order two
H_2 = { (1, 1), (1, b)}
H_3 = { (1, 1), (a^2, 1)}
H_4 = { (1, 1), (a^2, b)}

Subgroup of $C_4 \times C_2$ of order four
H_5 = { (1, 1), (1, b), (a^2, 1), (a^2, b) }
H_6 = { (1, 1), (a, 1), (a^2,1), (a^3,1) }
H_7 = { (1, 1), (a^2, 1), (a,b), (a^3 , b) }

Subgroup of $C_4 \times C_2$ of order eight

Table 2: Character table of C_4

G	1	a	a2	a3
χ_1	1	1	1	1
χ_2	1	1	-1	$-i$
χ_3	1	-1	$-i$	1
χ_4	1	$-i$	1	-1

Table 3: Character table of C_2

g	1	b
ψ_1	1	1
ψ_2	1	- 1

So that character table of $C_4 \times C_2$ will be

Table 4: Character table of $C_4 \times C_2$

g	(1,1)	(1,b)	(a,1)	(a2,1)	(a2,b)	(a2,b)	(a2,b)	(a3,b)
χ_1 ´ ψ_1	1	1	1	1	1	1	1	1
χ_2 ´ ψ_2	1	1	1	1	-1	-1	$-i$	$-i$
χ_3 ´ ψ_1	1	1	-1	-1	$-i$	$-i$	1	1
χ_4 ´ ψ_1	1	1	$-i$	$-i$	1	1	-1	-1
χ_1 ´ ψ_2	1	-1	1	-1	1	-1	1	-1
χ_2 ´ ψ_2	1	-1	1	-1	-1	1	$-i$	i
χ_3 ´ ψ_3	1	-1	-1	1	$-i$	i	1	-1
χ_4 ´ ψ_4	1	-1	$-i$	$-i$	1	-1	-1	1

Irr (C_4 ´ C_2/H_{-5}) = { χ_3 ´ ψ_3 }
Now, cd (C_4 ´ C_2/H_5) = {1}
\ |cd (C_4 ´ C_2 / H_5)| = 1

The derived length of H_5
H_5 = {(1,1), (1,b), (a^2,1), (a^2, b)}
[(1,b), (a^2,1)] = $(1,b)^{-1}$ $(a^2,1)^{-1}$ (1,b) (a^2,1)
= (1,b) (a^2,1) (1,b) (a^2,1)

= (1,1)

$[(1,b), (a^2,b)] = (1,b)^{-1} (a^2,b)^{-1} (1,b) (a^2,b)$
$= (1,b) (a^2,b) (1,b) (a^2,b)$
$= (1,1)$

$[(a^2,1), (a^2,b)] = (a^2,b)^{-1} (a^2,b)^{-1} (a^2,1) (a^2,b)$
$= (a^2,1) (a^2,b) (a^2,1) (a^2,b)$
$= (1,1)$

\ $H_5^1 = (1,1)$, $H_5 \supset H_5^1 = (1,1)$
\ the derived length of $H_5 = 1$
\ $dl(H_5) \leq |cd\ (C_4$ ´ $C_2/H_5)|$

In particular, H_5 is abelian

$(1,b)$ ´ $(a^2,1) = (a^2,b)$
$(a^2,1)$ ´ $(a,b) = (a^2,b)$
Also (a^2,a) ´ $(a^2$ ´ $b) = (1,b)$
(a^2,b) ´ $(a^2$ ´ $1) = (1,b)$

Theorem 3: Isaacs and Greg Theorem [10]
Let N Δ G and suppose that $|cd\ (G/N)| = 2$. If N is solvable, then dl (N) = 2.
Verification:

Let S_3 ´ C_2 be a finite group of order 12; where

$S_3 = \{ \begin{pmatrix}123\\123\end{pmatrix}, \begin{pmatrix}123\\132\end{pmatrix}, \begin{pmatrix}123\\213\end{pmatrix}, \begin{pmatrix}123\\231\end{pmatrix}, \begin{pmatrix}123\\312\end{pmatrix}, \begin{pmatrix}123\\321\end{pmatrix} \}$
(1) (23) (12) (123) (132) (13)

$C_2 = \{1,a\}$, $a^2 = 1$

S_3 ´ C_2 = { ((1), 1), ((23), 1), ((12),1), ((123),1),
((132),1), ((13) ,1), ((1), a), ((23), a), ((12),a), ((123), a),
((132), a) ((13), a)}

Order of each element in S_3 ´ C_2

((1),1)	–	1
((23),1)	-	2
((12),1)	-	2
((123),1)	-	3
((132),1)	-	3
((13),1)	-	2
((1), a)	-	2
((23), a)	-	2
((12), a)	-	2
((123), a)	-	6
((132), a)	-	6
((13), a)	-	2

1 element subgroup

H_1 = {(1), 1)}

2 elements subgroup

H_2 = { ((1), 1), ((2,3), 1)}
H_3 = { ((1), 1), ((12), 1)}
H_4 = { ((1), 1), ((1,3), 1)}
H_5 = { ((1), 1), ((1), a)}
H_6 = { ((1), 1), ((23, a)}
H_7 = { ((1), 1), ((12), a)}
H_8 = {((1), 1), ((13), a)}

3 elements subgroup

H_9 = {((1), 1), ((123), 1), ((132), 1) }

4 elements subgroup

H_{10} = {((1), 1), ((23), 1), ((1), a), ((123), 1)}
H_{11} = {((1),1), (12), 1), ((1), a), (12), a)}
H_{12} = {((1), 1), ((13), 1) ((1),a), ((13), a)}

6 elements subgroup

H_{13} = {((1), 1), ((123), 1), ((132), 1), ((123), a), ((132), a) ((1), a)}
H_{14} = {((1), 1), ((23), 1), ((12), 1), ((132), 1), ((13), 1), ((123), 1)}
H_{15} = {((1), 1), ((23), a), (12), a), ((132), 1), ((13), a), ((13), a)}

12 element subgroup

H_{16} = {((1), 1), ((23), 1), ((12), 1), ((123), 1), ((132), 1), ((13), 1)}
((1), a), ((23), a), ((12), 1), ((123), a), ((132), a), ((13), a)}

Testing for normal subgroup, we get

H_9 = {((1), 1), ((123), 1), ((132), 1)} to be a normal subgroup of S_3

The character table of S_3 is given below:

S_3 has 3 conjugacy classes (1), (12) and (132) with 3 irreducible characters

Table 5: Character table of S_3

Representative	(1)	(12)	(132)
Class Size	1	3	2
|CG(g)|	6	2	3
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Table 6: Character table of C_2.

G	1	a
Class Size	1	1
|CG(g)|	2	2
ψ_1	1	1
ψ_2	1	- 1

So that we have the following for the character table of S_3 ´ $C_{2.}$

Table 7: Character table of S_3 ´ C_2

Representatives	((1),1)	((1),a)	((12),1)	((12),a)	((132),1)	(132 a,)
χ_1 ´ ψ_1	1	1	1	1	1	1
χ_2 ´ ψ_2	1	1	-1	-1	1	1
χ_3 ´ ψ_1	2	2	0	0	-1	-1
χ_1 ´ ψ_2	1	-1	1	-1	1	-1
χ_2 ´ ψ_2	1	-1	-1	1	1	-1
χ_3 ´ ψ_2	2	-2	0	0	-1	1

So Irr (S_3 ´ C_2 / H_9) = { χ_3 ´ ψ_1 , χ_3 ´ ψ_2 }
and |cd(S_3 ´ C_2 / H_9)| = {2,2}
\ |cd (S_3 ´ C_2 / H_9)| = 2
To test for the solvability of H_9
H_9 = {((1,1), ((123), 1), ((132), 1)}
Picking two elements ((123), 1) and ((132), 1), we get

H_9 = [$\begin{pmatrix}123\\231\end{pmatrix}$, 1) ($\begin{pmatrix}123\\312\end{pmatrix}$, 1)]

= ($\begin{pmatrix}231\\123\end{pmatrix}$, 1) ($\begin{pmatrix}312\\123\end{pmatrix}$, 1) ($\begin{pmatrix}123\\231\end{pmatrix}$, 1) ($\begin{pmatrix}123\\312\end{pmatrix}$, 1)

= ($\begin{pmatrix}231\\123\end{pmatrix}$, 1) ($\begin{pmatrix}312\\123\end{pmatrix}$, 1) ($\begin{pmatrix}231\\123\end{pmatrix}$, 1)

= ($\begin{pmatrix}231\\123\end{pmatrix}$, 1) ($\begin{pmatrix}123\\231\end{pmatrix}$, 1)

= ($\begin{pmatrix}231\\123\end{pmatrix}$, 1)

= ((1), 1)

Since the commutator of H_9 terminte at ((1), 1): it implies that H_9 is solvable.

\ H_9 ⊃ H_9^1 = {1}

So that the derived length is 1

(the derived length which is less than 2 satisfy the condition of the theorem.