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MARKOV MODELS FOR THE ANALYSIS OF DYNAMICAL SYSTEMS

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ABSTRACT

Most real world situations involve modelling of physical processes that evolve with time and space, especially those exhibiting high variability. Such events that have to flow with time or space are called dynamical systems. The mathematical notions of a dynamical system serves to depict the flow of causation from past into future (Kalman 1960). In this study, Markov model which is a signal model based on the Markovian property with state space approach was adopted for the analysis of dynamical systems. The Nigerian monetary exchange rate data was used in the application with the use of R statistical software package. The study incorporated the Chapman-Kolmogorov equation in the construction of absolute limiting distribution of the system via the state variables. The procedure gives an easy and effective means of analysing complex and time varying dynamical systems. The study showed that the Nigerian monetary exchange rate is ergodic with stationary probability distribution.

Key words: Markov process, Random Walk, Dynamical system, Chapman-Kolmogorov equation, Signal model, Ergodicity

INTRODUCTION

Most real world situations involve modelling of physical processes that evolve with time and space, especially those exhibiting high variability. Such events that have to flow with time or space are called dynamical system. Dynamical system is a means of describing how one state develops into another state over the course of time. Technically, a dynamical system is a smooth action of the real, or the integers on another object usually a manifold. the mathematical notions of a dynamical system serves to depict the flow of causation from past into future (Kalman 1960). These models are considered and used in physics, engineering, financial and economic forecasting as an abstract summary of experimental data.

The general forms for dynamical systems are:

for continuous time

$$\dot{X} = f(X,t) \quad , \quad t \in \mathbb{R} \tag{1}$$

for discrete time

$$X(t+1) = f[X(t),t] \quad , \quad t \in \boldsymbol{Z} \tag{2}$$

X is the dynamical variable and $t,$ the time parameter.

If the right hand sides of equations (1) and (2) are nonlinear functions, then we have nonlinear dynamical system. The range of behaviours available to nonlinear systems is much greater than that for linear systems. These systems are characterized by a lot of uncertainties which need to be well captured.

According to Meiss (2007), a dynamical system consists of an abstract phase space or state space, whose coordinates describe the state at any instant and a dynamical rule that specifies the immediate future of all state variables, given only the present values of these same state variables. Mathematically, a dynamical system is described by an initial value problem. The implication is that there is a notion of time and that a state at one time evolves to a state or possibly a collection of states at a later time. These states can be ordered by time, and time can be thought of as a single quantity. Dynamical systems are mathematical objects used to model physical phenomena whose state (or instantaneous description) changes over time. These models are used in financial and economic forecasting, environmental modelling, medical diagnosis, industrial equipment diagnosis, and a host of other applications.

Modelling and estimation of dynamical systems has been of great interest among researchers. Nonlinear Dynamical System behaviours are in different forms which can range from very simple periodic solutions to complicated "chaotic" behaviour (Devaney, 1989). Mowery (1965) and Neal (1968) used the methods of least squares to minimize the error in nonlinear estimation. Other researchers used Gaussian probability density functions in modelling and estimating nonlinear systems. Lainiotis (1971) used Gaussian probability density functions to predict the most likely values of the state variables based on the current values of the output and the covariance of the state estimation error. Hall et al (2012) used Gaussian processes as a predictive model in modeling nonlinear dynamical systems. In most dynamical systems which describe processes in engineering, physics and economics, stochastic components and random noise are included. The stochastic aspects of the models are used to capture the uncertainty about the environment in which the system is operating and the structure and parameters of the models of physical processes being studied (Shali, 2012). Most dynamical phenomena in nature therefore, can be regarded as stochastic processes whose future behaviour can be modelled on the present state and not on the past. By treating them as such, meaningful results both in the theory and application may be obtained. Such processes are referred to as Markov processes.

Markov processes are random (or stochastic) processes whose future behaviour cannot be accurately predicted from its past behaviour and which involve random chance or probability. Markov processes are probabilistic models for describing data with a sequential structure. A Markov process is useful for analyzing dependent random events; that is, events whose likelihood depends on what happened last. A coherent mathematical theory of Markov processes in continuous time was first introduced by Kolmogorov (Dynkin, 2006). Important contributions to this class of stochastic processes were made by Feller (1971). A Markov process is a stochastic system for which the occurrence of a future state depends on the immediately preceding state. The transition probability is therefore a conditional probability for the next state given the current state. In the theory of Markov processes, it is usually a question of dealing, not with a single random function, but with a family of such functions, corresponding to all the possible initial instant of time and all the possible initial states.

The theory of Markov processes has developed rapidly in recent years. The properties of the trajectories of such processes and their infinitesimal operators have been studied, and intimate connections discovered between the behaviour of the trajectories and the properties of the differential equations corresponding to the process (Dynkin, 2006). According to Aoki (1994, 1996), dynamic behaviour can be modelled as discrete time or continuous time Markov chains. Markov chain is a Markov processes with finite state spaces. Time evolution of the probabilities of states of Markov chains can be described by accounting for probability flows into and out of the states (Davies, 1993; Durrett, 1991; Eckstein and Wolpin, 1989). These can be handled by Chapman-Kolmogorov equations in stochastic processes (Whittle, 1992; Bailey, 1984; Tomasz *et al*., 1999; Stark and Woods, 1986). The Chapman-Kolmogorov equations supply both necessary and sufficient conditions for the existence of transition densities. Akintunde *et al* (2008) proposed a modified Chapman-Kolmogorov equation which was seen to be efficient. This work therefore considered the analysis of nonlinear dynamical systems using the basic concept of Markov process which are those of a state and state transitions with the Chapman-Kolmogorov equation. The Markov model is a signal model based on the Markovian property which implies that given the present state, the future of a system is independent of its past. A Markov process is in a sense the probabilistic analog of causality and can be specified by defining the conditional distribution of the random process (Agwuegbo et. al (2014).

MATERIALS AND METHODS

Let X_t be the realization from a dynamical system. The state of a dynamical system $\{X_t\}$ is the probability distribution of the states, and knowledge of the distribution determines uniquely probabilistically how the system evolves with time in the future. By the central limit theorem, X_t may be considered to follow a normal distribution. A convenient way to understand the central limit theorem is by defining the independent identically distributed random variables $X_1, X_2, \ldots$ with a common distribution F as a random walk. One of the simplest stochastic processes is a simple random walk. A mathematical formulation of a path that consists of a succession of the random steps is by defining it as a random walk.

Random walks explain the observed behaviour of the random process and thus serve as a fundamental model for the recorded stochastic activity. Simple Random walk can be defined as the sum of a sequence of random variables. The simple random walk process arises in many ways. The random walk can be thought of as the path of a drunkard, walking along a long road who randomly takes either one step forward or one step backward at regularly spaced times. An alternative picture of random walk involves the motion of a particle which inhabits the set of integers and which moves at each step either one step to the right with probability p or one step to the left with probability q. The directions for the different steps are independent of each other. The random walk in a discrete case has three (3) values -1,0,+1 which constitute the generating function as a trinomial distribution. Then the defining condition of the random walk is

$$S_n = S_0 + \sum_{t=1}^{n} X_t \qquad (3)$$

Where S_0 and X_t are independent and identically distributed random variables each taking the values-1 with probability q or the value 0 with probability r or the value +1 with probability p. If $S_0 = 0$, then (3) becomes

$$S_n = \sum_{t=1}^{n} X_t \qquad (4)$$

and S_n is the distribution of the partial sums of the random variables. The distribution of S_n for finite n can therefore be determined, given the assumption about X_t as

$$X_t = \begin{cases} +1 & with\ probability\ p \\ -1 & with\ proba \qquad bility\ q \\ 0 & with\ probabilityb\ r \end{cases} \qquad (5)$$

For the description of the system, the sequence X_t can further be be classified as a Gaussian random walk to enumerate all possible states of the system, where X_t are jointly normally distributed. The importance of the normal distribution is due largely to the central limit theorem.

In our case $\{X_t\}$ is the realization from the stochastic process. Given a stochastic process, we can assign according to some rule, to each of its function X_t a new function Y_t known as transformation of X_t into Y_t .In this work, let $\{Y_t\}$ be the set consisting of the relative changes as the dynamical system evolve through time. That is

$$Y_t = X_t - X_{t-1} \qquad (6)$$

This provides a transformation of the original realization of the system.

We define a trinomial distribution for the various values of Y_t .Let Z_t be the state of Y_t . Z_t equals +1 the with probability p if the value of Y_t is positive, Z_t is -1 with probability q if the value of Y_t is negative and Z_t is 0 with probability r if the value of Y_t is zero. That is

$$Z_t = \begin{cases} +1 & with\ probability\ p \\ -1 & with\ probability\ q \\ 0 & with\ probabilityb\ r \end{cases} \qquad (7)$$

Binomial and trinomial processes are simple examples for general random walks that is stochastic processes $\{Y_t; t \geq 0\}$ satisfying

$$Y_t = Y_0 + \sum_{k=1}^{t} Z_k, \qquad t = 1,2,\ldots \qquad (8)$$

where Y_0 is independent of $Z_1, Z_2, \ldots$ which are independent identically distributed (i.i.d.) is referred to as a random walk. The increments which have a distribution of a real valued random variable, Z_k can take a finite or countable infinite number of values; but it is also possible for Z_k to take values out of a continuous set. Such random walks can be constructed by assuming identically, independent and normally distributed increments. By the properties of the normal distribution, it follows that X_t is $N(\mu t, \sigma^2 t)$ -distributed for each t . If $X_0 = 0$ and $Var(Z_1)$ is finite, it holds approximately for all random walks for t large enough. Random walks are processes with independent increments (Franke *et al,* 2004) and processes which are also Markov processes.

The system can be in any of the enumerable sequence of states $Y_1, Y_{2,\ldots}$ .The sequence of random variables $Y_1, Y_{2,\ldots}$ forms a discrete time Markov chain if for all $n(n = 1,2,\ldots)$ and all possible values of the random variables, one has (for $i_1 < i_2 < \cdots < i_n$) that

$$P(Y_n = j | Y_1 = i_1, Y_2 = i_2, \ldots, Y_{n-1} = i_{n-1})$$

$$= P(Y_n = j | Y_{n-1} = i_{n-1}) \quad (9)$$

The expression on the right hand side of the equation (9) is referred to as one step transition probability and gives the conditional probability of making a transition probability from state $E_{i_{n-1}}$ at step $n-1$ to state E_j at nth step in the process. If it turns out that the transition probabilities are independent of n, it turns out to what is referred to as a homogeneous Markov chain which is defined as

$$p_{ij} = P(Y_n = j | Y_{n-1} = i_i) \quad (10)$$

which gives the probability of going to state E_j on the next step, given that it is currently at state E_i. These chains in (10) are such that their transition probabilities are stationary with time and the probability n step into the future depends only upon n and not upon the current time; it is expedient to define the $n-step$ transition probabilities as

$$p_{ij}^{(n)} = P(Y_{m+n} = j | Y_m = i) \quad (11)$$

From the Markov property given in (9), it is easy to establish the following recursive formula for calculating $p_{ij}^{(n)}$:

$$p_{ij}^{(n)} = \sum_k p_{ik}^{(n-1)} p_{kj} \qquad n = 2,3,\dots \quad (12)$$

Generalizing the homogeneous definition for the multistep transition probability given in (11), one can now define

$$p_{ij}(m,n) = P(Y_n = j | Y_m = i) \quad (13)$$

Which gives the probability that the system will be in state E_j at step n, given that it was in state E_i at step m, where $n \geq m$.

Equation (13) may be expressed as the sum of probabilities for all these mutually exclusive intermediate states, that is,

$$p_{ij}(m,n) = \sum_k P(Y_n = j, Y_q = k | Y_m = i) \qquad (14)$$

For $m \leq q \leq n$. (14) must hold for any stochastic process since one is considering all mutually exclusive and exhaustive possibilities. From the definition of conditional probability, one may rewrite (14) as

$$p_{ij}(m,n) = \sum_k P(Y_q = k | Y_m = i) P(Y_n = j | Y_m = i, Y_q = k) \qquad (15)$$

Invoking the Markov property and observing that

$$P(Y_n = j | Y_m = i, Y_q = k) = P(Y_n = j | Y_q = k)$$

And applying this to (15) and making use of the definition in (13), then one arrives at

$$p_{ij}(m,n) = \sum_k p_{ik}(m,q) p_{kj}(q,n) \qquad (16)$$

For $m \leq q \leq n.$ Equation (16) is called the Chapman-Kolmogorov equation for discrete time Markov processes. If (16) is a homogeneous Markov chain, then (11) will have the relationship $p_{ij}(m,n) = p_{ij}^{(n-m)}$.

In a completely analogous way, a continuous time Markov chain can define as

$$p_{ij}(s,t) = P[Y(t) = j | Y(s) = i] \qquad (17)$$

Where $Y(t)$ is the position of the particle at time $t \geq s.$ The Chapman-Kolmogorov equation for three successive instant of time $s \leq u \leq t$ for the continuous time chain is given as

$$p_{ij}(s,t) = \sum_k p_{ik}(s,u)\, p_{kj}(u,t) \qquad (18)$$

In matrix form, (18) is written as

$$H(s,t) = H(s,u)H(u,t) \qquad s \leq u \leq t \qquad (19)$$

which is the transition probability matrix of the Markov chain
Further, (19) is identified as a limit and can be defined as

$$\frac{\partial H(s,t)}{dt} = H(s,t)Q(t) \quad s \leq t \qquad (20)$$

(20) is called the forward Chapman-Kolmogorov equation for the continuous time. Considering the case where the continuous time Markov chain is time homogeneous or stationary, the transition probability function is estimated as $p_{ij}(t)$

$$p_{ij}(t) = P(Y(t) = j|Y(0) = i) \qquad (21)$$

The sufficiency of (21) is the consequences for both Markov and stationarity assumptions. The forward Chapman-Kolmogorov equation for the stationary process is defined as

$$p_{ij}(t+s) = P(Y(t+s) = j|Y(0) = i)$$

$$= \sum_k P(Y(t+s) = j, Y(t) = k|Y(0) = i)$$

$$= \sum_k P(Y(t+s) = j|Y(t) = k, Y(0) = i)\, P(Y(t) = k|Y(0) = i)$$

$$= \sum_k P(Y(t+s) = j|Y(t) = k)\, P(Y(t) = k|Y(0) = i)$$

$$= \sum_k P(Y(s) = j|Y(0) = k)\, P(Y(t) = k|Y(0) = i)$$

$$= \sum_k p_{kj}(s)p_{ik}(t)$$

$$p_{ij}(t+s) = \sum_k p_{ik}(t)p_{kj}(s) \qquad (22)$$

(22)is for a continuous process. The Chapman-Kolmogorov equation indicates that the transition probability can be decomposed. And in matrix form can be defined as

$$H(s,t) = H(s)H(t) \qquad (23)$$

Stationary Analysis and Absolute Probabilities

At the long run the system settles down to a condition of stationary statistical equilibrium in which the state occupation probabilities are independent of the initial conditions. Dynamical system can be considered as a memory-less process because the randomness involved had introduced a noise component which accounted for the irregular behaviour of the system. As a result of this, the system may be considered to satisfy the Markovian property. The absolute probability distribution converges to a limiting distribution independent of the initial distribution, and the chain is said to be ergodic. Ergodicity defines the limiting values or steady state probabilities such that the actual limiting distribution, if it exists can be determined quite easily. The equation for the steady state distribution can be obtained as:

$$\lim_{n\to\infty} P^{(n)} = P \qquad (24)$$

$$\lim_{n\to\infty} P\{Y_n = j | Y_0 = i\} = \lim_{n\to\infty} P\{Y_n = j\} = p_j \qquad (25)$$

If a Markov chain is ergodic, then the limiting distribution is stationary and

$$\pi P = \pi \qquad (26)$$

where $\pi = (\pi_i, i = 1,2 \dots n)$ and

$$\sum_i \pi_i = 1 \qquad (27)$$

RESULTS AND DISCUSSION

The dynamics discussed in this work was used to model the Nigerian monetary Foreign exchange rates. Monthly Data of the Exchange of Nigerian Naira to US dollar covering a period of thirty four (34) years, from 1980 to 2013 was considered and a direct analysis was accomplished by the use of R statistical software.

Exploratory Data Analysis

The data were explored with the use of some descriptive statistics (Table 1), stationarity or unit root tests (Table 2) and graphs (Fig. 1, 2 and 3). Table 1 gives the descriptive statistics values. Fig. 1 shows that there are no outliers in the data set and Fig. 2 indicates that there is a dramatic jump in the system. The realisation plot (Fig. 3) shows an indication that the variable is non-stationary hence a realization from a nonlinear dynamical system. It revealed a dramatic jump in 1999 and that the monthly exchange rate of Nigerian Naira to United States Dollar was highly volatile. Tables 2 which consist of the Augmented Dickey-Fuller and the Phillips-Perron Tests for Stationarity also confirmed that the series is non-stationary. The dramatic jump in 1999 can be attributed to the political transition from military to democratic government which brought about some changes in policies. Fig. 4 indicates continuous sample paths of the state spaces. It shows that the Nigerian monetary exchange rate follows a random walk model.

Table 1: Descriptive Statistics of the Exchange Rate Data

Statistics	Exchange Rate Data
Length	396
Minimum	0.5314
Maximum	164.6000
1st Quartile	4.286
3rd Quartile	127.2
Mean	60.56
Median	21.89
Sum	23981.95
SE Mean	3.0684
Variance	3728.347
Standard deviation	61.06
Skewness	0.4093
Kurtosis	-1.6229

Table 2: Augmented Dickey-Fuller and Phillips-Perron Tests for Stationarity of the data

Unit Root Test	Hypothesis	Test Statistics	p-value	Decision
Augmented Dickey-Fuller	H0: The Series is Non Stationary H1: The Series is Stationary	-2.1434	0.5167	Accept Ho
Phillips-Perron	H0: The Series is Non Stationary H1: The Series is Stationary	-13.8437	0.3357	Accept Ho

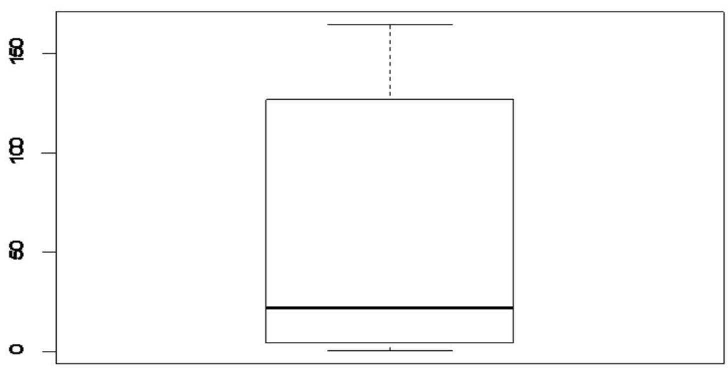

Figure 1: Boxplot of the Exchange Rate Data

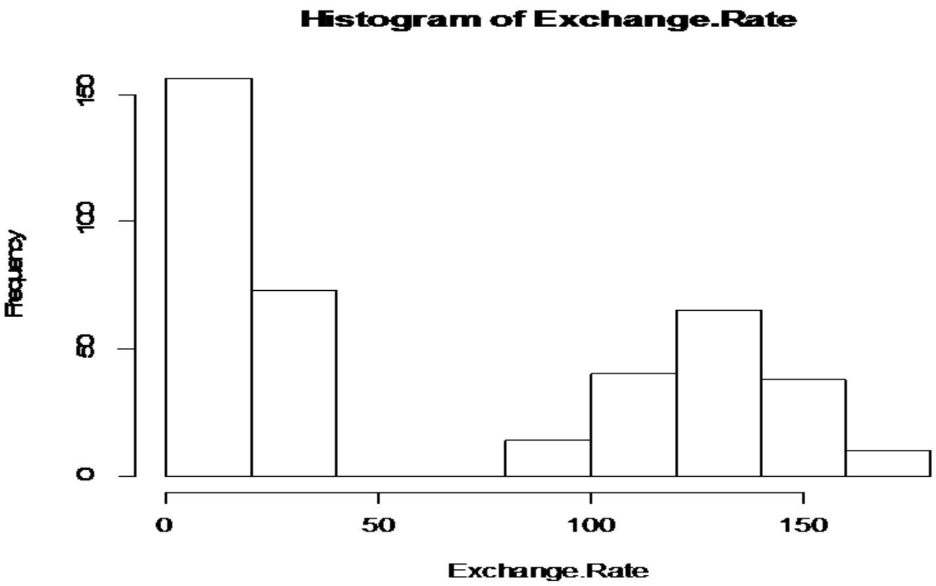

Figure 2: Histogram of the Exchange Rate Data

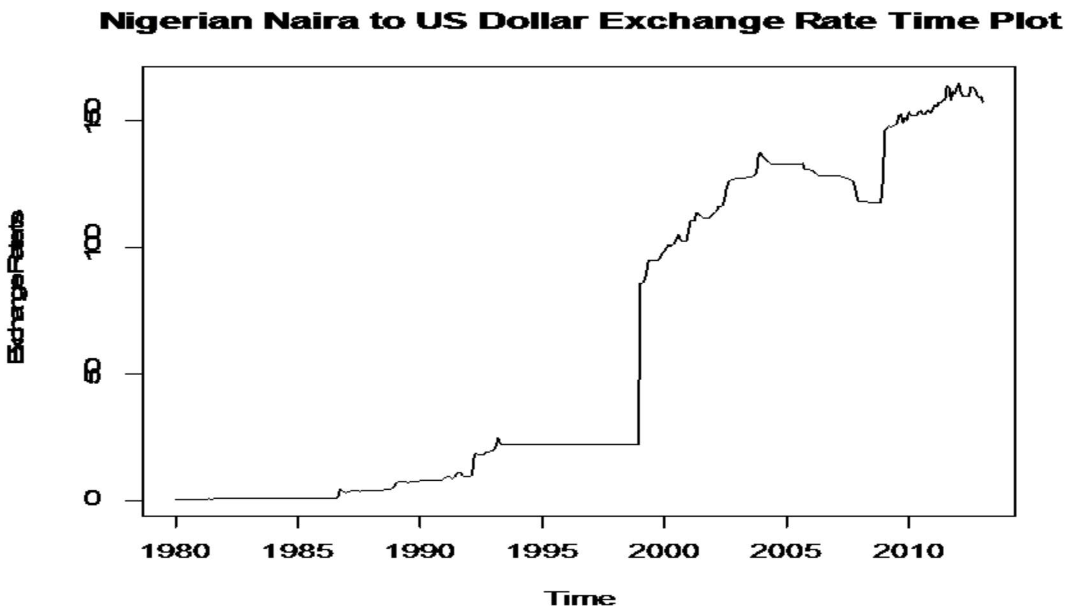

Figure 3: Realisation plot of the Exchange rate of Nigerian Naira to US Dollar

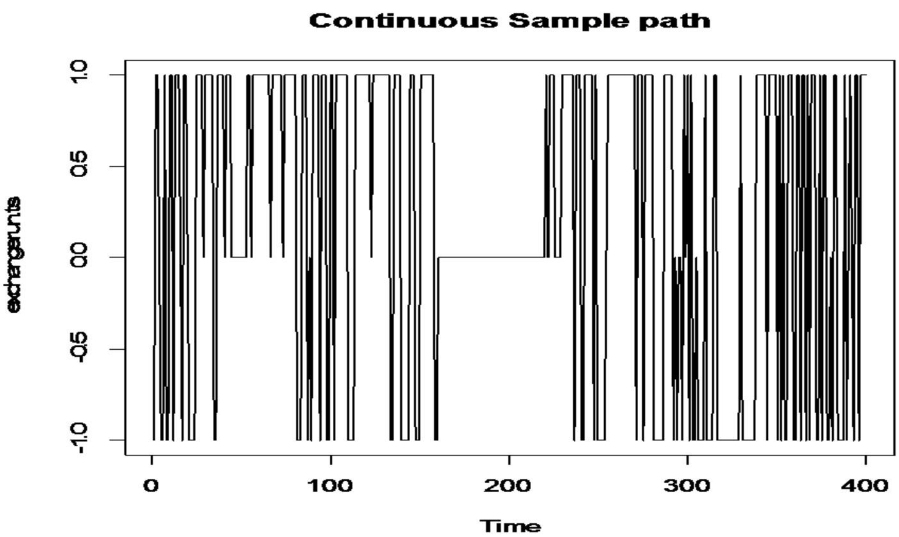

Figure 4: Continuous Sample path of Nigerian Exchange rate to US Dollar

Stationary Analysis

The generating function can therefore be given as trinomial distribution from the Random Walks. This gives the initial probability of the state of the process as

$$p^{(0)} = (0.4500, \quad 0.1000, \quad 0.4500)$$

and the transition probability matrix given as

$$P_{ij} = \begin{pmatrix} 0.5794 & 0.0561 & 0.3645 \\ 0.0761 & 0.7826 & 0.1413 \\ 0.2158 & 0.0632 & 0.7211 \end{pmatrix}$$

With the Chapman-Kolmogrov equation, the stationary probability distribution of the system can be obtained. Using the Chapman – Kolmogorov relation

$$P(x_n = j) = p^{(0)}P^n \qquad (28)$$

The absolute Probabilities are given as

$$p_n = p_i^0 P^n$$

$$= p^{(n)} \qquad (29)$$

where P^n is the n-step transition probability matrix given as

$$P^n = P_{ij}^{(n)}$$

$$= p^{n-1}P \qquad (30)$$

and p_i^0 is the initial probability distribution.

Using the Chapman-Kolmogorov equation recursively, the distribution converges to the fixed probability vector which equals the absolute or stationary probability distribution of the system at time, t_n. This is as shown in Table 3. Therefore the steady state probability distribution of the system which does not depend on the initial probability distribution is

$$p^{(n)} = (0.2916, \quad 0.2179, \quad 0.4915) \qquad (31)$$

Table 3: Absolute Probabilities of the State of the Systems at time, t_n

n	$p_1^{(n)}$	$p_2^{(n)}$	$p_3^{(n)}$
1	0.3655	0.1319	0.5027
2	0.3303	0.1555	0.5143
3	0.3142	0.1727	0.5132
4	0.3059	0.1853	0.5090
5	0.3012	0.1943	0.5047
6	0.2982	0.2009	0.5012
7	0.2962	0.2056	0.4985
8	0.2949	0.2090	0.4965
9	0.2939	0.2115	0.4950
10	0.2932	0.2133	0.4940
11	0.2927	0.2146	0.4932
12	0.2924	0.2155	0.4927
13	0.2921	0.2162	0.4923
14	0.2920	0.2167	0.4920
15	0.2918	0.2171	0.4918
16	0.2917	0.2173	0.4917
17	0.2917	0.2175	0.4916
18	0.2917	0.2177	0.4916
19	0.2916	0.2179	0.4915
20	0.2916	0.2179	0.4915
21	0.2916	0.2179	0.4915
22	0.2916	0.2179	0.4915
23	0.2916	0.2179	0.4915
24	0.2916	0.2179	0.4915
25	0.2916	0.2179	0.4915

This implies that the value of Nigerian Naira will at the long run reduce as compared to the US Dollar with a probability of 49%. This dynamics can be better viewed with the use of high frequency data such as weekly, daily or even hourly data of the exchange rate.

CONCLUSION

Dynamical systems are very common and are considered to be stochastic processes by virtue of their own mechanism. The study considered dynamical system as a random process with independent increments and as a result can be modelled with Markov chains models. The system was therefore viewed as a Random Walk from the state spaces of the Markov process. The Chapman-Kolmogorov equation gave a straight forward iteration which converges to the stationary distribution of the process. The study shows that the Nigerian exchange rate to US Dollar is ergodic.

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