ISSN:
Print - 2277 - 0593
Online - 2315 - 7461

Journal of Natural Science, Engineering and Technology

ON WEAK AMENABILITY OF RESTRICTED SEMIGROUP ALGEBRA AND SEMIGROUP ALGEBRA ON RESTRICTED SEMIGROUP

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ABSTRACT

We studied weak amenability of restricted semigroup algebra $l_r^1(S)$ and semigroup algebra $l^1(S_r)$, on restricted semigroup S_r. We give condition for the restricted semigroup algebra to be commutative for every inverse semigroup S. Some classes of inverse semigroups such as semilattice, Clifford and Brandt semigroup are used to characterize a weakly amenable restricted semigroup algebra. In particular, we show that for a Clifford semigroup $S = \cup_{i=1}^{n} G_i$ and the Brandt semigroup $S = M^0(G, I, n)$, the weak amenability of semi- group algebra $l^1(S)$, restricted semigroup algebra $l^1(S)$, and semigroup algebra $l^1(S_r)$, on restricted semigroup S_r are equivalent. In general, the necessary and sufficient conditions for weak amenability of restricted semigroup algebra and semigroup algebra $l^1(S_r)$, on restricted semigroup S_r are given.

Keywords: semigroup, restricted semigroup, semigroup algebra, weakly amenable.
A.M.S (2010) Mathematics Subject Classification: 46H20, 46H10, 46H25.

DOI:

INTRODUCTION

Bade et al. (1987) introduced the notion of weak amenability for commutative Banach algebras while Johnson (1988) considered the same notion for arbitrary Banach algebra and in (Johnson 1991) showed that the group algebra $L^1(G)$ of a locally compact group G is always weakly amenable. The authors in (Bade et al. 1987) defined a weakly amenable Banach algebra A as one in which every continuous derivation from it into a commutative Banach A-bimodule is necessarily zero. This definition was extended by the author in (Gronbaek 1989) by giving some characterizations of weakly amenable Banach algebra in terms of the splitting of an admissible sequence. Blackmore (1997) particularly showed that if S is a completely regular semigroup, then the semigroup algebra, $l^1(S)$ is weakly amenable and that if S is commutative, then the weak amenability of $l^1(S, \omega)$ for any weight ω implies the weak amenability of $l^1(S)$. He further gave some conditions for which the semigroup algebra $l^1(S)$ is weakly amenable. Haagerup (1983) showed that C*-algebra is always weakly amenable.

Massoud Amini and Alireza Medghalchi (2006) introduced a class of Banach algebra called the restricted semigroup algebra and the authors in (Mewomo and Ogunsola 2016), Mohammad and Massoud (2010) and Sahleh and Grailo (2015) respectively studied the character amenability, amenability and module amenability of this class of Banach algebra. This study is motivated by the introduction of this class of Banach algebra and the need to investigate the notion of weak amenability that has not been studied on this class of Banach algebra. In this work, we investigate both the commutativity and non-commutativity of different classes of discrete inverse semigroup to establish the weak amenability of restricted semigroup algebra and that of the semigroup algebra l^1S_r on restricted semigroup S_r .

PRELIMINARIES AND DEFINITIONS

First we recall some standard notions. For further details see (Dales 2000).

Let A be an algebra and let X be an A-bimodule. A linear map D : A → X is called a derivation if:

D(ab) = D(a).b + a.D(b) (a, b ∈ A).

This derivation is inner if for x ∈ X, there is a map δx : A → X such that

$\delta_x(a)$: a.x − x.a (a ∈ A).

Clearly, every inner derivation is a derivation.

Let A be a Banach algebra and let X be a Banach A-bimodule. The Banach algebra A is amenable if the first cohomology group of A with coefficient in dual module X', of X is trivial i.e $H^1(A, X') = \{0\}$.

If A' = X', then the Banach algebra A is said to be weakly amenable as $H^1(A, A') = \{0\}$.

We recall from (Johnson 1972) that the first cohomology group of A with coefficients in X is the quotient space: $H^1(A, X) = Z^1(A, X)/N^1(A, X)$, where $Z^1(A, X)$ is the space of all continuous derivations from A into X and $N^1(A, X)$ is the space of all inner derivations from A into X.

The Banach algebra A is n-weakly amenable if $H^1(A, A^{(n)}) = \{0\}$ where n ∈ N and $A^{(n)}$ is the nth conjugate space of A.

A is permanently weakly amenable if A is n-weakly amenable for every n ∈ N. See (Dales et al. 1998).

Let S be a semigroup.

Let s ϵ S. An element s* ∈ S is called an inverse of s if ss*s = s and s*ss* = s*.

An element s ∈ S is called regular if there exists t ∈S with sts = s.

An element s ∈ S is called completely regular if there exists t ∈ S with sts = s and ts = st.

S is called regular if each s ∈ S is a regular element.

S is called completely regular if each s ∈ S is a completely regular element. Completely regular semigroups are those which can be regarded as the disjoint unions of their maximal subgroups (Blackmore 1997).

S is called an inverse semigroup if S is regular and every element in S has a unique inverse.

An element p ∈ S is called an idempotent if $p^2 = p$; the set of idempotents of S is denoted by E(S). E(S) is also defined as a commutative subsemigroup of an inverse semigroup S.

S is called a semilattice if it commutes and E (S) = S.

An inverse semigroup S is called a Clifford semigroup if $ss^{-1} = s^{-1}s$ for each s ∈ S.

Let S be a Clifford semigroup and let s ∈ S. Then s $\in G_{ss^{-1}}$ and hence S is a disjoint union of the groups G_p (p ∈ E(S)). That is

S = Up ∈ E(S)G_p where G_ps are the maximal subgroups of S.

GENERAL RESULTS

In this section, we prove some general results which are useful in establishing our main results on restricted semigroup algebra and semigroup algebra, l^1S_r on restricted semigroup Sr.

Proposition 3.1

Let S be an inverse semigroup which is a union of groups. Then the following are equivalent:

(i)Every idempotent in S is commutative.

(ii) $l^1(S)$is weakly amenable.

Proof:

(i) This follows from [Clifford and Preston (1961), Theorem 4.11].

(ii) If S is commutative, then by [Gronbaek (1989), Corollary 2.8], $l^1(S)$ is weakly amenable. We recall that a Banach algebra A is said to be graded over a semigroup S if we have closed subspaces A_s for each $s \in S$ such that $A = l^1 - \bigoplus s \in S\ A_s$; $A_sA_t \subset A_{st}$ $(s,t \in S)$.

For the case in which S is a finite semilattice, each A_s is a closed subspace of A.

Proposition 3.2

Let S be a finite semilattice and let A be a Banach algebra graded over S. Then A is weakly amenable if and only if each A_s is weakly amenable.

Proof:

The result clearly follows from [Gronbaek (1989), Theorem 2.7].

Let A be a Banach algebra and let *I* be a non-empty set. Let the set of $I \times I$ matrices (a_{ij}) with entries in A be denoted by $M_I(A)$ such that $||(a_{ij})|| = \sum_{i,j\in I} ||a_{ij}|| < \infty$. Then $M_I(A)$ with the usual matrix multiplication is a Banach algebra that belongs to the class of l^1-Munn algebras. It is an easy verification that the map $\theta : M_I(A) \rightarrow M_I(C) \otimes A$ defined by $\theta((a_{ij})) = \sum_{i,j\in I} E_{i,j} \otimes a_{ij}$ $((a_{ij})\ M_I(A))$, is an isometric isomorphism of Banach algebras, where (E_{ij}) are the matrix units in $M_I(C)$.

Proposition 3.3

Let A be a weakly amenable Banach algebra and J a non-empty set. Then Mj(A) $\sim= A \otimes$ Mj(C) is weakly amenable if and only if Mj(C) is weakly amenable.

Proof:

It is well known that Mj(A) $\sim=$ A $\otimes$Mj(C). If Mj(C) is weakly amenable, then by Proposition 2.6 (Gronbaek 1989), Mj(A) is weakly amenable. Conversely, if Mj(A) is isomorphic to A $\otimes$ Mj(C) and Mj(A) is weakly amenable, then A $\otimes$ Mj(C) is weakly amenable and so Mj(C) is weakly amenable.

Let η be a relation on a commutative inverse semigroup S defined as $a\eta b$ $(a, b \in S)$ if and only each of the elements a and b divides some power of the other. More details are given in [14]. Clearly the relation η on S is a congruence relation and by Theorem 4.12 (Mohammad and Massoud 2010), S/η is the maximal semilattice homomorphic image of S.

Let S be an inverse semigroup and $p \in E(S)$. We set $G_p = \{s \in S : ss^{-1} = s^{-1}s = p\}$. where s^{-1} denotes the inverse of s. Then G_p is a group with identity p and G_p contains other subgroup of S with identity p. Thus G_p is called maximal subgroup of S at p.

If S is a semilattice, it then suffice to say that S/η is isomorphic to G_p.

Proposition 3.4

Let $S = U_p \in {}_{E(S)}G_p$ be a Clifford semigroup. Then $l^1(S)$ is weakly amenable if and only if $l^1(G)$ is weakly amenable.

Proof:

Clearly $l^1(S) = l^1(G)$ $(G_p = G)$. If $l^1(G)$ is weakly amenable, then it is immediate from Proposition 3.1 that $l^1(S)$ is weakly amenable. Conversely, suppose $l^1(S)$ is weakly amenable. From $l^1(G) \subset l^1(G_p)$ $l^1(S)$ we can conclude that $l^1(G)$ is weakly amenable.

MAIN RESULTS

In this section, we shall consider the weak amenability properties of restricted semigroup algebra $l^1(S)$ and semigroup algebras,

l^1S_r on restricted semigroup Sr using different characterizations. For details on restricted semigroups and restricted semigroup algebras see (Johnson 1991) and Mewomo 2011). For any inverse semigroup S, the restricted product of elements s and t of S is st if s*s = tt* and undefined, otherwise. The set S with this product forms a discrete groupoid and if we adjoin a zero element 0 to this groupoid with 0* = 0, we get an inverse semigroup denoted by Sr, with the multiplication:

$$s \bullet t = \begin{cases} st = s*s = tt* \\ 0 \quad otherwise, \end{cases}$$

(s, t ∈ S ∪ {0}) which is called the **restricted semigroup** of an inverse semigroup S.
It is clear that E(Sr) = E(S) ∪ {0}.
Suppose S is a ∗-semigroup, given a Banach space $l^1(S)$ with the usual l^1- norm, we set f˜ (x) = $\overline{f(x)}$ and define the following multiplication on $l^1(S)$.

$$(f \bullet g)(s) = \sum_{s*s=tt*} f(st)g(t*) \qquad (s \in S)$$

Then ($l^1(S)$, •,) with the l^1-norm is a Banach ∗-algebra denoted by $l_r^1(S)$ called the restricted semigroup algebra of S.
Since S is discrete, $l_r^1(S)$ is a discrete semigroup algebra.
$l_r^1(S)$ = {f : S → C : $\sum_{s\in S} |f(s)| < \infty$}, | | = $f||_1$ = $\sum_{s\in S} |f(s)|$
The module action of on is shown as follows:
(f, g.λ) = (f.g, λ) , (f, λ.g) = (g.f, λ) (f, g ∈ l^1S_r λ ∈ $l_r^\infty(S)$).

Clearly, $l_r^\infty(S)$ is a Banach-$l_r^1(S)$ bimodule.
For a restricted semigroup S_r of an inverse semigroup S, l^1S_r is called the semigroup algebra on restricted semigroup S_r.

In the following Lemma, we show the commutativity of restricted semigroup algebra.

Lemma 4.1
Let S be an inverse semigroup. Then the restricted semigroup algebra $l_r^1(S)$ on an inverse semigroup S, is commutative if and only if S is a semilattice.
Proof :
By the definition of restricted semigroup algebra $l_r^1(S)$, we have for each f, g ∈ $l_r^1(S)$ f • g(u)= $\sum f(ut)g(t*)$, if u*u=tt* and 0 otherwise.
Let S contain idempotent elements which commute i.e for s = ut then su* =t, t*=s*u.
So $\sum f(s)g(s*u)$ = f • g(u).
Similarly, g • f(u)= $\sum g(ut)f(t*)$
if u*u=tt* and 0 otherwise.
With s = ut, $\sum g(s)f(s*u)$ = g • f(u). This shows that $l_r^1(S)$, •) is a commutative Banach algebra.
Conversely, suppose ($l_r^1(S)$, •) is commutative, then for f, g ∈ ($l_r^1(S)$ •), (f • g)s = (g • f)s (s ∈ S).
By definition of the restricted product on ($l_r^1(S)$, •), (f • g)s = $\sum f(st)g(t*)$
if s*s = tt* otherwise it is zero.
Similarly, (g • f)s = $\sum g(st)f(t*)$ with the same condition. This implies that
$\sum f(st)g(t*) = \sum f(t*)g(st)$.
Hence we have st = t* (i)
Now suppose t* = s*u and substituting in (i) gives st = s*u.
Thus s = s*ut* = s*ust = s*sut. This implies that s = ut. and hence S is a semilattice.

Example 4.2
Let (S, +) be an abelian semigroup such that s + s = t + t = s + t = s = t for all s, t ∈ S. Clearly S is a semilattice. Then by Lemma 4.1, $l^1(S)$ is weakly amenable.

Corollary 4.3
Let S be a finite semilattice . Then the following are equivalent:
(i)$l^1(S)$ is commutative.
(ii) l^1S_r is weakly amenable.

Proof :
(i) This is Example 4.2.
(ii) Suppose $l^1(S)$ is commutative. We recall that $S_r = S \cup \{0\}$ and $S \subset S_r$. Clearly $l_1(S) \subset l^1S_r$. Since $l^1(S)$ can be embedded in l^1S_r, it then follows from [Gronbaek (1989), Corollary 2.9] that l^1S_r is weakly amenable.
In the following theorem, we characterize restricted semigroup algebras using the concept of derivation.

Theorem 4.4
Let $A = l_r^1(S)$ be a commutative restricted semigroup algebra on an inverse semigroup S and let A' be the dual space of Banach algebra A. Then A is weakly amenable if every continuous derivation $D : l_r^1(S) \to A'$ is inner for every commutative Banach - $l_r^1(S)$ -bimodule.
Proof :
Let $D : l_r^1(S) \to A'$ be a continuous derivation for every Banach -A-bimodule.
Clearly D is a bounded linear map. To show that D is an inner derivation, let $f, g \in l_r^1(S)$ and $u \in S$, we have
$D(f \bullet g)(u) = D\,(\sum f(ut)g(t*)) = D\,((\sum f(ut).(\sum g(t*))$
$= \sum f(ut)D(\sum g(t*)) + D(\sum f(ut))(\sum g(t*))$
Suppose S contains idempotent elements which commute. By Lemma 4.1, then we have
$= \sum f(s)D(\sum g(us*)) + D(\sum f(s))(\sum g(s*u))$
$= \sum f(s)D(\sum g(uu*t)) + D(\sum f(s))(\sum g(u*t*u))$
$= \sum f(s)D(\sum g(t*)) + D(\sum f(s))(\sum g(t*))$
$= f\,D(g) + D(f\,)g.$
Clearly D is a derivation. Now to show that D is inner.
Let $D(f\,) = \delta_\lambda f = f.\lambda - \lambda.f$ for each $\lambda \in l^\infty(S)$.
Thus we have :
$(f.g, \lambda) = (f, D(g)) + (g, D(f\,))$
$= (f, g.\lambda - \lambda.g) + (g, f.\lambda - \lambda.f\,) = 0.$
It then follows that $D = \delta_\lambda = 0$ and so we conclude that D is an inner derivation. The proof is complete.
Let G be a group, a Brandt semigroup S over a group G with index set J is the semigroup consisting of elementary $J \times J$ over $G \cup \{0\}$ and a zero matrix $\{0\}$.
We write $S = \{(g)_{ij} : g \in G, i, j \in J\} \cup \{\{0\}\}$, with multiplication given by:
$(g)_{ij}\,(h)_{kl} = (gh)_{il}$ if j=k and 0 if j $\neq$ k.
The Brandt semigroup is an inverse semigroup.
For an inverse semigroup S, it was shown in [9] that the restricted semigroup $S_r = S_i \cup_{i \in I}$ for Brandt semigroup S_i with $S_i \cap S_j = S_iS_j = \{0\}$, if i $\neq j$.
The next result shows weak amenability of Brandt semigroup algebras while the subsequent ones give the necessary and sufficient conditions for $l_r^1(S)$ and $l^1(S_r)$.

Theorem 4.5

Let $S = M^o(G, I, n)$ be a Brandt semigroup. Then the semigroup algebra, $l^1(S_r)$ on a restricted semigroup S_r , is weakly amenable if and only if the semigroup algebra $l^1(S)$ on a Brandt semigroup S is weakly amenable.

Proof :

For a Brandt semigroup S_i of finite index, $S_r = \cup_{i\in I} S_i$. It then suffice to say that $l^1 = (S_r) \cup l^1(S_i) = l^1(S)$. Thus if $l^1(S)$ is weakly amenable, $l(S_r)$ is also weakly amenable. The converse follows easily.

Proposition 4.6

If S is an inverse semigroup, then $C\delta_0$ is an essential closed ideal of $l^1(S_r)$.

Proof :

By [Mohammad and Massoud (2010), Lemma 2.5], $C\delta_0$ is a closed ideal of $l^1(S_r)$ Let $I = C\delta_0$ and $A = l^1(S_r)$

Let $[I : A] = span\{s.f - f.s | s \in I, f \in A\}$. Clearly, $[I : A] = I$. Since $[I : A] = I = \bar{I}^2$ [4, Theorem 3.2], then $C\delta_0$ is an essential ideal of $l^1(S_r)$.

Theorem 4.7

Let $l^1(S_r)$ be a weakly amenable Banach algebra. Then $l_r^1(S)$ is weakly amenable if and only if $l^1(S_r)$ has an essential closed ideal.

Proof :

Suppose $C\delta_0$ is an essential closed ideal of $l^1(S_r)$. By Proposition 2.2 (Gronbaek 1989), $C\delta_0$ is weakly amenable. By [Massoud and Alireza (2006), Theorem 3.7], the restricted semigroup algebra $l^1(S)$ is isomorphic to the quotient space, $l^1(S_r)/C\delta_0$. Then it clearly follows that $l_r^1(S)$ is weakly amenable.

Corollary 4.8

Let $l_r^1(S)$ be a restricted semigroup algebra on an inverse semigroup S. Then $l_r^1(S)$ is weakly amenable if and only if $l(S_r)$ is weakly amenable.

Proof :

By Proposition 4.6, $C\delta_0$ is an essential ideal of $l^1(S_r)$. Using Theorem 4.7, the result clearly follows.

Proposition 4.9

Let $S = \cup_{i=1}^n G_i$ be a Cliford semigroup. Then the following are equivalent.

(i) $l_r^1(S)$ is weakly amenable.

(ii) $l^1(S_r)$ is weakly amenable.

(iii) $l^1(S)$ is weakly amenable.

Proof :

$\Rightarrow$ *(ii)* By Corollary 4.8, if $l_r^1(S)$ is weakly amenable so is $l^1(S_r)$.

$\Rightarrow$ *(iii)* $S_r = S \cup \{0\}$. Let $S_i = G_i \cup \{0\}$, $i=1,2,\ldots n$, *then* S_i is a Brandt semigroup with group G_i. Thus $S_r = \cup n\ S_i$ with $S_i \cap S_j = S_i S_j = \{0\}$.

If $l^1 S_r$ is weakly amenable, then by Theorem 4.5, $l^1(S)$ is weakly amenable.
⇒ (i) The result follows from Theorem 4.5 and Corollary 4.8.
We recall that $S_r = S \cup \{0\}$. We have $l^1(E_r) = (l^1(E \cup \{0\})^\bullet)$ as a subalgebra of $l^1(S_r)$. Hence $l^1(E_r) \subseteq l^1(S_r)$. Now suppose S_r is a finite semilattice. Let $A = l^1(S_r)$ and let $A_{sr} = l^1(E_r)$. Hence $A = l^1 \bigoplus A_{sr}$: $A_{sr}\, A_{tr} \subset A_{str}$, s_r, $t_r \in S_r$. Clearly, each $A_s r$ is a closed subalgebra of A.
The following result is an analogue to Proposition 3.2.

Proposition 4.10
Let S_r be a finite semilattice and let A be a Banach algebra graded over S_r. Then A is weakly amenable if and only if each $A_s r$ is weakly amenable.

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(Manuscript received: 21st October, 2021; accepted: 12th October, 2022).