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RESOLUTION OF EXTREMISATION PROBLEM USING DUALITY PRINCIPLE

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ABSTRACT

The use of duality principle for characterizing solution of general optimization problem posed in the Hilbert space was considered. The existence and uniqueness of solution are guaranteed by formulating the minimum problem in a dual space. Furthermore, the solution is shown to be aligned.

Keywords: Extremisation problem, Duality Principle, Optimum solution, Hilber space.

INTRODUCTION

The essence of optimization abstraction (Bamigbola *et al.,* 2005) is for ease of characterizing the solution of optimization problems. The modern theory of optimization in normed linear space is largely centered about the interrelations between a space and its corresponding dual. Duality plays a role analogous to the inner product in Hilbert space. The main duality principle is that the primal problem has an optimal solution if and if the dual problem has an optimal solution. Let us consider the extension theorem extending projection solution of minimum norm problems to arbitrary normed spaces.

Theorem 1.1

According to (Madox, 1988) let x be a real linear normed space and *p* a continuous sub linear functional on *X*. let *f* be a linear functional defined on a subspace *M* of *X* satisfying $f\ (m) \leq p(m)$ for all m € *M*. Then there is an extension F of f on M such that $F\ (x) \leq p\ (x)$ *on* X.

Definition 1.1

Given a vector *x* in a Hilbert space *H* and a subspace *M* in *H*. We wish to find the vector *m* closest to *x* in the sense that it maximizes $\|x- m\|$ is called a minimum norm problem. If M is a closed subspace of Hilbert space, there is always a unique solution to the minimum problem and the solution satisfies orthogonality condition.

Furthermore, we introduced the following two theorems, which centre on the equivalence of two extremisation problems: one formular in a normed space X and the other in its dual x'. We remark here that the minimum norm problems have been found useful extensively in approximation theory, Estimation theory, etc.

Theorem 1.2

According to (Luenberger 1969) let *x* be an element in a real normed linear space X. Let *d* denote its distance from the subspace *M*.

Then:

$$d = \inf_{m \in M} \| x - m \| = \max_{\substack{\|x'\| \le 1 \\ x' \in M^{\perp}}} \langle x, x' \rangle$$

where the minimum on the left is achieved for $M|_0 \in M^{\perp}$. If the supremum on the right is achieved for some $x_0 \in M^{\perp}$, then $x^1 - m^1_0$ said to be aligned with $x_0 - m_0$.

Corollary 1.1

According to Luenberger (1969) let x be an element of a real normed linear space X and let M be a subspace of X. A vector $m_0 \in M$ satisfies $\|x - m_0\| \le \|x - m\|$ for all $m \in M$ if and only if there is a non-zero $x' \in M^{\perp}$ aligned with $x - m_0$.

Theorem 1.3

According to Luenberger (1969) let M be a subspace in a real normed space X. Let $x' \in X$ be a distance d from $M^{\perp}$. Then:

$$d = \min_{\substack{m' \in M^{\perp} \\ \|x\| \le 1}} \|x' - m'\| = \sup_{x \in M} \langle x, x' \rangle$$

where, the minimum on the left is achieved for $m_0' \in M^{\perp}$. If the supremum on the right is achieved for some $x_0 \in M$, then $x' - m_0'$ is said to be aligned with x_0.

The Dual of C^p [a,b]

Given the interval [a,b] and $1 \le p < \infty$ is the collection of all real valued function that are p-times continuously differentiable on [a,b]. clearly C^p [a,b] is a vector space. We show that for each $x \in$ **C^p [**a,b**],** the following

$$\|x\| = \max_{t \in J} |x(t)| + \max_{t \in J} |x^1(t)| + \ldots + \max_{t \in J} |x^{(p)}(t)| = \sum_{K=0}^{p} \max |x^{(k)}(t)|$$

where: $J = [a,b]$ defines as norm

$$\|\alpha x\| = \max |\alpha x(t)| + \max |\alpha x^1(t)| + \ldots + \max |\alpha x^{(p)}(t)| = |\alpha| \{ \max |x(t)| + \max |x^1(t)| + \ldots + \max |x^{(p)}(t)| \}$$

$$= |\alpha| \, \|x\|$$

If, x, y € $\mathbf{C}^p$ then for any k = 0,1 . . .,p

$| x^{(k)}(t) + y^{(k)}(t) | \le | x^{(k)}(t) + y^{(k)}(t) | \le | \max x^{(k)}(t) | + \max | y^{(k)}(t) |$

and also

$$\max | x^{(k)}(t) + y^{(k)}(t) | \le | \max x^{(k)}(t) | + \max | y^{(k)}(t) |$$

consequently $$\|x + y\| = \sum_{K=0}^{p} \max | x^{(k)}(t) + y^{(k)}(t) |$$

$$\le \sum_{K=0}^{P} \max | x^{(k)}(t) | + \sum_{K=0}^{p} y^{(k)}(t) | = \|x\| + \|y\|$$

[Here, $x^{(o)}(t) = x(t)$].

Hence, $\mathbf{C}^p\,[a,b]$ is a normed space.

We next establish the Riesz representation theorem for elements f of the dual of the real linear space $C^p[a,b]$. We define a functional f on $C^p[a,b]$ by:

$$f(x) = \int_a^b \{x(t) + x'(t) + \cdots + x^{(p)}(t)\} dt.$$

Definitely, f is linear and bounded with norm

$$\|f\| = b - a$$

$$|f(x)| = \left| \int_a^b \{x(t) + x'(t) + \cdots + x^{(p)}(t)\} dt \right|$$

$$\le (b-a)\left\{\max_{t\in J} |x(t)| + \max_{t\in J} |x'(t)| + \cdots + \max_{t\in J} |x^{(p)}(t)|\right\}$$

$$= (b-a)\|x\|$$

taking the supremum over all x of norm 1, we obtain $\|f\| \le b - a.$

To get $\|f\| \ge b - a$, we choose $x(t) = x'(t) = \cdots = x^{(p)}(t) = 1$

$$\|f\| \ge \frac{|f(x)|}{\|x\|} = |f(x)| = \int_a^b dt = b - a$$

Theorem 2.1 (Modified Reisz Representation Theorem)

Let *f* be a bounded functional on $X = C^p[a,b]$. Then, there is a function *g* of bounded variation on $[a,b]$ such that for all $x \in X$,

$$f(x) = \int_a^b \{x(t) + x'(t) + \cdots + x^{(p)}(t)\} dg(t)$$

and such that the norm of *f* is the total variation of *g* on $[a,b]$ defines a bounded linear function on *X* in this way.

Proof

Let *B* be the space of bounded variation on $[a,b]$ with the norm of an element $x \in B$ defined as

$$\|x\|_B = \sup_{t \in J} |x(t)|$$

The space $C^p[a,b]$ can be considered as a subspace of *B.* Thus, if *f* is a bounded linear functional on $X = C^p[a,b]$, there is, by Hahn-Banach theorem, a linear functional *F* on *B* which is now an extension of f and has the same norm. For any $q \in [a,b]$, we define the function u_q by $u_a = 0,$ and

$$u_q(t) = \begin{cases} 1 & if \quad a \le t \le q; \\ 0 & if \quad q \le t \le b. \end{cases}$$

for $a < q \le b$ Clearly, each $u_q \in B.$

Define $g(q) = F(u_q)$ and show that g is of bounded variation on $[a,b]$. For this purpose, let $a < t_0 < t_1 < t_2 < \ldots < t_n = b,$ be a finite partition of $[a,b]$. Denoting

$$\lambda_i = \operatorname{sgn}\, (g(t_i) - g(t_{i-1})),$$

we have,

$$\sum_{i=1}^{n} |(g(t_i) - g(t_{i-1}))| - \sum_{i=1}^{n} \lambda_i [(g(t_i) - g(t_{i-1}))]$$

$$= \sum_{i=1}^{n} \lambda_i \left[\left(F\left(U_{t_i}\right) - F\left(U_{t_{i-1}}\right)\right)\right]$$

$$= F \sum_{i=1}^{n} \lambda_i \left[\left(\left(U_{t_i}\right) - \left(U_{t_{i-1}}\right)\right)\right]$$

$$\leq \|F\| \left\| \sum_{i=1}^{n} \lambda_i \left[\left(\left(U_{t_i}\right) - \left(U_{t_{i-1}}\right)\right)\right] \right\| = \|f\|$$

and

$$\|F\| = \|f\|$$

and

$$\left\| \sum_{i=1}^{n} \lambda_i \left[\left(U_{t_i}\right) - \left(U_{t_{i-1}}\right)\right] \right\| = 1$$

Hence, g is of bounded variation with

$$TV(g) = V(g) \leq \|f\|$$

Next, we derive a representation for *f* on *X*. Let

$$z = \sum_{i=1}^{n} x(t_{i-1})\left[U_{t_i} - U_{t_{i-1}}\right]$$

where t_i is again a finite partition of $[a,b]$ and $x \in X$. Then,

$$\|z - x\|_B = \max_{t_{i-1} \leq t \leq t_i} \left| x(t_{i-1}) - x(t_i) \right|$$

which goes to zero as the partition are made arbitrarily fine (since $C^P[a,b]$ is uniformly continuous). Using the continuity of *F*,

$$F(z) \to F(x) = f(x),$$

but,

$$F(z) = \sum_{i=1}^{n} x(t_{i-1})[g(t_i) - g(t_{i-1})]$$

and by the definition of Steiltjes integral,

$$F(z) \to \int_a^b x(t)dg(t)$$

$$f(x) \to \int_a^b x(t)dg(t)$$

Now, it is a standard property of the Steiltjes integral that

$$\left|\int_a^b x(t)g(t)\right| \le \|x\|V(g)$$

for each $x \in C^P[a,b]$,
hence,

$$\|f\| \le V(g)$$

on the other hand,

$$\|f\| \ge V(g)$$

using Reisz representation and consequently,

$$\|f\| = V(g)$$

It should be noted that the function *g* in Theorem 2.1 is not unique. To remove the ambiguity, we introduce a subspace of $BV[a,b]$ called the normalized space of functions of bounded variation denoted by $\overline{B}\overline{V}[a,b]$ consisting of functions which vanish at the point $t = a$ and are continuous from the right on $t = a$ in $[a,b]$.

Characterizing Optimum Solution

In this section, we consider the use of the above results. Three steps can be identified in connection with the minimum norm problem:

1. The use of alignment property of the space to characterize the optimum solution
2. The existence of solution guaranteed by formulating the norm problem in a dual space.
3. Checking to see if the dual problem is easier to solve than the primal.

Many problems amendable to the theory of Section 1 are mostly naturally formulated as finding the vector of a minimum norm in a linear variety rather than finding the best approximation on a subspace. A standard problem to this kind arising in several con texts is to find an element of minimum norm satisfying a finite number of linear constraints. To guarantee existence of solution, let us consider *x*' € x' and express the constraints in the form:

Y$_1$, *x*' = c$_1$
Y$_2$, *x*' = c$_1$
…………
…………
…………
Y$_n$, *x*' = c$_n$ *y*$_1$ € x'

If x' is any vector satisfying the constraints, we have

$$d = \min_{\langle y_i, x \rangle = c_i} imize \|x'\| = \min_{m' \in M'} imize \|\bar{x}' - m'\|$$

where: M denotes the space generated by the y_i's . But from Theorem 1.3.

$$d = \min_{m' \in M'} imize \|\bar{x}' - m'\|$$

any vector in M is of the form

$$x = \sum_{i=1}^{n} a_i y_i$$

, since M is finite dimensional

$$d = \min_{\langle y_i, x' \rangle = c_i} \|\bar{x}'\| = \max_{\|x\| \le 1} \langle x, \bar{x}' \rangle = \max_{\|x\| \le 1} c^* a$$

Alignment Property

Definition 4.1

A vector $x' \in X''$ is said to be aligned with a vector $x \in X$ if

$$\langle x, x' \rangle = \|x\| \|x'\|$$

where: X' denote the dual of *X*. We remark that the alignment is a relationship between two vectors spaces.

Corollary 4.1

Let $y_i \in X,\ i = 1,2,\ldots,n$ is consistent; and suppose the system of linear equations $\langle y_i, x' \rangle = c_i,\ i = 1,2,\ldots,n$ is consistent.
Then,

$$\min_{x' \in K} \|x'\| = \max_{\|x\| \le 1} c^* a$$

Furthermore, the optimal x' is aligned with the optimal *x*.

Proof

Let $K = \{x' \in X' : \langle y, x' \rangle = c_{i,} i = 1,2,...,n\}$ be non empty.

Suppose $x' \in K, \quad \|x'\| = d.$ Then, $\langle x, x' \rangle \le \langle y_i, x' \rangle = d.$ Since $x' \in K$

However,

$$\langle x, x' \rangle \le \|x\|\|x'\| = d$$

Thus,

$$\langle x, x' \rangle \le \|x\|\|x'\|$$

and x' is aligned with *x.*

Existence and Uniqueness of Solution

Theorem 1.3 guarantees the existence of a solution to the minimum norm problem if the problem is appropriately formulated in the dual of a normed space. This is also reflected in Hahn Banach theorem which establishes the existence of certain linear functionals.

The optimal vector, if it exists, may not unique as the situation is likely to be more complex in arbitrary normed spaces since the equation for optional vector will generally be non-linear. Nevertheless, the key concept that quarantees uniqueness is the orthogonality condition and the principal result is analogous to the projection theorem.

Selection of the Relevant Problem

The transition from one problem to its dual results in significant simplification as it convert some complex infinite-dimensional problems to an easily handled finite dimensional problems (Ferreira, 1996). Even in the finite dimensional problems, the higher the dimensions, the higher the complexity.

CONCLUSION

Optimization involving extremisation problem over Hilber space is herein considered with the use of extension theorem and alignment property. The optimal solution is characterized by the space **C^p [*a,b*],** $1 \le p \le \infty$.

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